CES 4141 Stress analysis Fall 1998

Solution of Simultaneous Linear Equations

Matrices can be used as an organized way of presenting a set of coupled equations. Generally speaking, we seek a single unique solution that satisfies all the equations at the same time. Consider the equations below:

$$
\begin{aligned}
3X_1 + 5X_2 + 2X_3 &= 8 \\
2X_1 + 3X_2 - 1X_3 &= 1 \\
1X_1 - 2X_2 - 3X_3 &= -1
\end{aligned}
$$

These are referred to as <u>coupled linear equations</u>. *Coupled* because each equation has one or more terms in common with the others, X_1, X_2, X_3, so that a change in one of these variables will affect more than one equation. *Linear* because each equation contains only first order terms of X_1, X_2, X_3. For example, there are no terms like X_1^2, or $\sqrt{X_2}$, or $\log(X_3)$, or $1/(X_1X_2)$, etc.Using the rules of matrix multiplication, we can represent the above equations in matrix form:

$$
\underbrace{\begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & -3 \end{bmatrix}}_{\text{coefficient matrix A}}
\underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{\text{unknowns vector X}}
=
\underbrace{\begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}}_{\text{solution vector B}}
$$

We'll refer to the coefficient matrix as A, the vector of unknowns as X, and the solution vector as B. Note that we are seeking the unknowns vector X, <u>not</u> the solution vector B, which is known information.

There are several ways to solve for the values of the unknown vector X. Each method involves some manipulations to the coefficient matrix using algebraic rules, such that we create a new and equivalent problem in a more easily solvable form.

These manipulations involve the addition of multiples of one row to another. Since each row represents an equation, where the left hand side (l.h.s.) is equal to the right hand side (r.h.s.), adding one row to another results in an equivalent equation. For example, starting with the two equations:

$$X_1 + 5X_2 = 3 \tag{1}$$

$$-2X_1 - 3X_2 = 5 \tag{2}$$

their addition

$$X_1 + 5X_2 - 2X_1 - 3X_2 = 3 + 5 \tag{3}$$

will not change the equality, or the solution to X_1, X_2. This addition does not add any new information either, but it does present a new form of the old information, i.e.

$$-1X_1 + 2X_2 = 8. \tag{4}$$

We will use this technique of adding the equations together to recast the original problem into a form that is easier to solve.

Gaussian Elimination (method #1):

Let’s consider the three coupled linear equations given on the previous page. The original form looks like this:

$$\begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix}. \tag{5}$$

But what if we could recast the same problem to look like this?

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & -\frac{11}{3} & -\frac{11}{3} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{11}{3} \\ -4 \end{bmatrix} \tag{6}$$

This makes life easier, since there is less coupling between equations. In fact, X_3 can be solved immediately using the bottom equation ==> $X_3 = 2$. Now the result can be used to write the middle equation as $-\frac{11}{3}X_2 - \frac{11}{3}(2) = -\frac{11}{3}$ to get ==> $X_2 = -1$. Finally, the known values X_1, X_2 are used to solve for X_1 in the first equation to get ==> $X_1 = 3$. This easy solution process is called *back-substitution*, and is made possible by the lower triangle of zeros in the coefficient matrix, marked with a dashed triangular box.

Great, it would be nice if the problem looked like Eq. (6), but it actually looks like Eq. (5), so what now? Well, we can use a series of additions of the 3 equations in Eq. (5) to get it to look like Eq. (6). In a series of steps just like the above addition of Eqs (1), (2) to get Eq. (3) we'll reduce the coefficients $A(2, 1), A(3, 1), A(3, 2)$ to zero, in that order.

KEY: Whatever we do to the l.h.s. of an equation, we do to the r.h.s. so we don't change the problem.

Let's rewrite the matrix in Eq. (6) to get one matrix with both A and B in it:

$$\begin{array}{rcl} \text{l.h.s.} & \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 2 & 3 & -1 & 1 \\ 1 & -2 & -3 & -1 \end{array}\right] & \begin{array}{l} \leftarrow \text{pivot row} \\ \\ \text{r.h.s.} \end{array} \\ & \text{A} \qquad\quad \text{B} & \end{array} \tag{7}$$

<u>Step 1) - reduce A(2,1) to zero</u>

New Row 2 = (Row 1)(-2/3) + (Row 2)

Row 1 is called <u>pivot row</u> for this step. Some multiple of it is added to another equation, but the pivot row remains unchanged

$$\begin{array}{r} \\ \text{add} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 3\left(-\frac{2}{3}\right) & 5\left(-\frac{2}{3}\right) & 2\left(-\frac{2}{3}\right) & 8\left(-\frac{2}{3}\right) \\ 2 & 3 & -1 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{7}{3} & -\frac{13}{3} \end{bmatrix} \end{array} \tag{8}$$

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & -\frac{1}{3} & -\frac{7}{3} & -\frac{13}{3} \\ 1 & -2 & -3 & -1 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{New Row 2} \\ \\ \end{array} \tag{9}$$

<u>Step 2) - reduce A(3,1) to zero</u>

New Row 3 = (Row1)(-1/3) + (Row 3)

Row 1 is the <u>pivot row</u> again

Expanding this instruction like we did in Eq.(8), the result is

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & -\frac{1}{3} & -\frac{7}{3} & -\frac{13}{3} \\ 0 & -\frac{11}{3} & -\frac{11}{3} & -\frac{11}{3} \end{array}\right] \longleftarrow \text{New Row 3} \qquad (10)$$

Now we need to reduce A(3,2) to zero. If we added some multiple of Row 1, then A(3,1) would become non-zero. Instead, we’ll need to add some multiple of Row 2 to Row 3 to get a new Row 3.

Before we go on, let’s consider error reduction...

<u>error reduction - swap Rows 2 and 3</u>

If there were some numerical error in the computer storage of any coefficient, say the error from rounding off the -1/3 currently in spot A(2,2), then when we multiply Row 2 by some factor and add it to Row 3, we also multiply the error by that factor. If we can always multiply by some small number (less than 1), we can reduce the propagation of that round-off error. We can enforce this by making sure the lead coefficient (the pivot coefficient) in the pivot row has the largest absolute value between among itself and all the coefficients under it (the coefficients to be reduced to zero).

Since it does not matter what order I put the equations in to solve them, we will rearrange rows when we find the current pivot coefficient has a smaller absolute value than those beneath it. In the current example we have:

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & -\frac{1}{3} & -\frac{7}{3} & -\frac{13}{3} \\ 0 & -\frac{11}{3} & -\frac{11}{3} & -\frac{11}{3} \end{array}\right]$$

Row 2 will be the pivot row to eliminate A(3,2), which makes A(2,2) the pivot coefficient. Since $\left|-\frac{11}{3}\right| > \left|-\frac{1}{3}\right|$, we’ll swap Rows 2 and 3, and the new pivot coefficient will be largest:

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & -\frac{11}{3} \\ 0 & -\frac{1}{3} & -\frac{7}{3} & -\frac{13}{3} \end{array}\right] \tag{11}$$

Now we can continue with minimal round-off error propagation

<u>Step 3) - reduce A(3,2) to zero</u>

New Row 3 = (Row 2)(-1/11) + (Row 3)

Row 2 is the pivot row now

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & -\frac{11}{3} & -\frac{11}{3} & -\frac{11}{3} \\ 0 & 0 & -2 & -4 \end{array}\right] \longleftarrow \text{New Row 3} \tag{12}$$

Now let’s expand this to its full form with A, X, B in separate matrices

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & -\frac{11}{3} & -\frac{11}{3} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -\frac{11}{3} \\ -4 \end{bmatrix} \tag{13}$$

We now have exactly what we wanted, that is, starting with a problem defined as in Eq. (5), and finding some <u>equivalent</u> problem Eq. (6) with the desired lower triangle of zeros. (13) and (5) are equivalent since all we did was a series of steps where we added the same thing to both sides of a row.

Now a little back-substitution (the paragraph following Eq. (6)) gives us

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \tag{14}$$