Direct Stiffness - Beam Application

We've looked at creating the global stiffness matrix for a structure built entirely of truss elements. These elements only have two local degrees of freedom, axial motion at each end. In this section we'll extend the ideas we used for trusses to create beam elements, and then assemble a global stiffness matrix for frames. Then we'll look at special cases where we may use only portions of the full element stiffness matrix.

1. Overview

Just as we did for trusses, we will consider each whole beam member as a *finite element*, and each joint becomes a node. We'll determine the force-displacement relationship for each element separately, then combine each individual contribution to the whole structure in a *Global Structural Stiffness Matrix* KG. What is different from a truss? Each node now has axial, shear, and moment 'degrees of freedom', rather than just axial.

These notes will present the stiffness for each beam member at the local level, then transform them into the global coordinate system, then sum the contribution of all elements, then discuss solutions for displacements, reactions, and internal forces.

2. Element Stiffness Matrix In Local Coordinates

We look at a single beam member in its own local coordinate system. Consider the element to the right. The local coordinate system (x', y') will align with the member orientation, for any arbitrary orientation. The local displacement degrees of freedom (D.O.F.) are labeled as v1 - v6.

Now we can apply the stiffness by definition procedure to find the stiffness matrix for this arbitrary element. Holding v1=1, v2 - v6=0, we get the left column of the 6x6 stiffness matrix by application of the slope deflection equations. Holding v1=0, v2=1, v3 - v6=0 we get the second column, etc. This is illustrated on page 158 in Hoit. However, we here include the axial displacements, which will give us a 6x6 local element stiffness matrix instead of the 4x4 given on page 158 (which ignores axial).

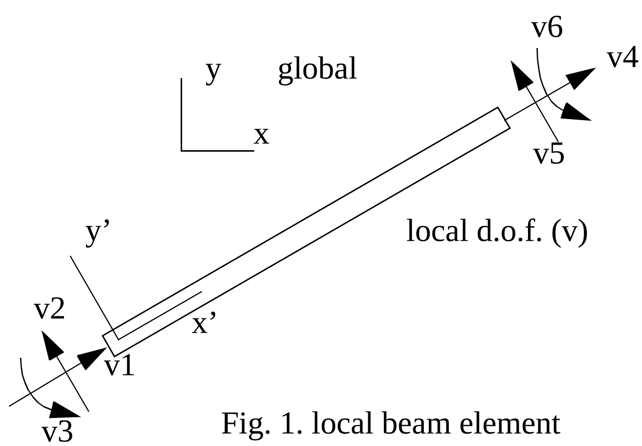


Fig. 1. local beam element

After the above procedure is completed by holding all six D.O.F. in turn to 1 and all others zero, we get the local stiffness matrix k given on the next page.

$$k = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (1)$$

Note that this will be the same no matter what the orientation of the element, since all quantities are with respect to the element local coordinate system, which will always be set up to lie along the axis of the member. Note also that the order is dependent on what order we chose to label v1 - v6, so we'll always label the d.o.f. of an element just as we did in Fig. 1.

Now we can relate the local forces to local displacements by

$$\begin{bmatrix} S1 \\ S2 \\ S3 \\ S4 \\ S5 \\ S6 \end{bmatrix} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ v3 \\ v4 \\ v5 \\ v6 \end{bmatrix}. \quad (2)$$

3. Displacement Transformation Matrix

Since a frame structure consists of many members of possibly many orientations, we'll want to transform all local displacements into a single uniform *global coordinate system*. The question is then, if we were to find the displacement at both ends of an element in terms of the local coordinates v1 - v6, what is that displacement in terms of the global coordinates (r1 - r6)? For the truss case at this point we were transforming 4 global degrees of freedom to 2 local degrees of freedom, and thus the transformation matrix a is a 2x4. In this case we are relating 6 global to 6 local degrees of freedom, so a will be a 6x6.

Refer now to the procedure we used in the truss development packet to find the transformation matrix. We moved an arbitrarily oriented member a unit value in each of the global degrees of freedom in turn, holding all other motions to zero, and finding what that does to the local component. We do the same here now and get the 6 equations below.

$$\begin{aligned} v1 &= r1*\cos\theta_x + r2*\cos\theta_y \\ v2 &= -r1*\cos\theta_y + r2*\cos\theta_x \\ v3 &= r3 \\ v4 &= r4*\cos\theta_x + r5*\cos\theta_y \\ v5 &= -r4*\cos\theta_y + r5*\cos\theta_x \\ v6 &= r6 \end{aligned} \quad . \qquad (3)$$

Using again $Lx = \cos\theta_x$, and $Ly = \cos\theta_y$, we re-write these equations in matrix form, and define the transformation matrix a

$$a = \begin{bmatrix} Lx & Ly & 0 & 0 & 0 & 0 \\ -Ly & Lx & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Lx & Ly & 0 \\ 0 & 0 & 0 & -Ly & Lx & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

Now we can relate global to local displacements in matrix form as

$$v = a*r \qquad (5)$$

Now we've found a way to relate local displacements to global displacements. We'll also want to do this for local and global forces.

4. Force Transformation Matrix

We can also identify the transformation matrix for the local/global force relationship. This time we'll get the global on the left hand side and the local on the right hand side by applying a unit force in local coordinates and finding the resultant global components. Just as was the case for the truss element, the resultant is the transpose of the a transformation matrix we found in Eq. 4. We can then relate local to global forces by

$$R = a^T*S \qquad (6)$$

5. Element Stiffness Matrix In Global Coordinates

Now we'll use the transformation matrices we just derived to find the stiffness matrix for a single element in terms of global coordinates. The local stiffness matrix will remain a 6x6.

We'll repeat the process we used for truss elements, since the specific element in question does not matter. Start with

$$S = k*v \tag{7}$$

Transform local displacements v to global displacements r by substituting Eq. 5 into Eq. 7 to get

$$S = k*a*r \tag{8}$$

Pre-multiply both sides by a^T, the left hand side becomes Eq. 6, and the final expression is

$$R = a^T*k*a*r \tag{9}$$

We have the displacements and forces in terms of global coordinates now. The stiffness matrix is still local, and is pre and post-multiplied by a^T and a. Let's separate this component and rename:

$$Ke = a^T*k*a \tag{10}$$

This gives us a final expression of

$$R = Ke*r \tag{11}$$

Let's write out and multiply through Eq. 10. Performing the matrix multiplications gives us:

$$,Ke = \begin{bmatrix} \frac{AE}{L}L^2x + \frac{12\mathrm{EI}}{L^3}L^2y & \left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{-6\mathrm{EI}Ly}{L^2} & -\left(\frac{AE}{L}L^2x + \frac{12\mathrm{EI}}{L^3}L^2y\right) & -\left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{-6\mathrm{EI}Ly}{L^2} \\ \left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{AE}{L}L^2y + \frac{12\mathrm{EI}}{L^3}L^2x & \frac{6\mathrm{EI}Lx}{L^2} & -\left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & -\left(\frac{AE}{L}L^2y + \frac{12\mathrm{EI}}{L^3}L^2x\right) & \frac{6\mathrm{EI}Lx}{L^2} \\ \frac{-6\mathrm{EI}Ly}{L^2} & \frac{6\mathrm{EI}Lx}{L^2} & \frac{4\mathrm{EI}}{L} & \frac{6\mathrm{EI}Ly}{L^2} & -\frac{6\mathrm{EI}Lx}{L^2} & \frac{2\mathrm{EI}}{L} \\ -\left(\frac{AE}{L}L^2x + \frac{12\mathrm{EI}}{L^3}L^2y\right) & -\left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{6\mathrm{EI}Ly}{L^2} & \frac{AE}{L}L^2x + \frac{12\mathrm{EI}}{L^3}L^2y & \left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{6\mathrm{EI}Ly}{L^2} \\ -\left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & -\left(\frac{AE}{L}L^2y + \frac{12\mathrm{EI}}{L^3}L^2x\right) & -\frac{6\mathrm{EI}Lx}{L^2} & \left(\frac{AE}{L} - \frac{12\mathrm{EI}}{L^3}\right)LxLy & \frac{AE}{L}L^2y + \frac{12\mathrm{EI}}{L^3}L^2x & \frac{-6\mathrm{EI}Lx}{L^2} \\ \frac{-6\mathrm{EI}Ly}{L^2} & \frac{6\mathrm{EI}Lx}{L^2} & \frac{2\mathrm{EI}}{L} & \frac{6\mathrm{EI}Ly}{L^2} & \frac{-6\mathrm{EI}Lx}{L^2} & \frac{4\mathrm{EI}}{L} \end{bmatrix} \tag{12}$$

6. Assembly of Global Stiffness Matrix

Before going on, let's review what we have derived. Eq. 11 relates the displacements to forces, all in global coordinates, for a single element of arbitrary orientation. The orientation of the individual member is accounted for in the *Global element stiffness matrix* Ke.

What's left to do? We now have the contribution of a single beam element. For a structure with multiple members, we assemble a Ke for *each* member in the frame, then add them each into a *global structural stiffness matrix* that covers every degree of freedom in the entire structure. We'll call that KG. KG will be a square matrix with as many rows and columns as there are total degrees of freedom, *frozen and unfrozen*. Let's do this by example.

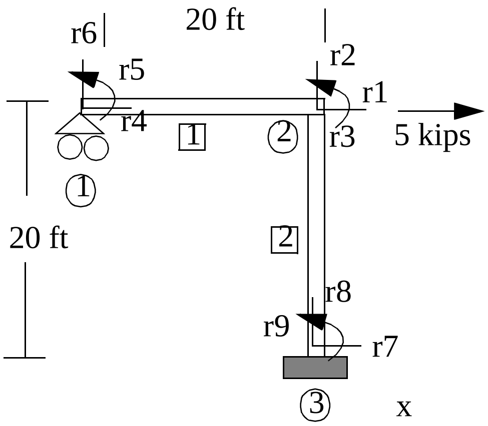


Fig. 2. Frame example

<u>Example 1: Assembly of KG for a truss structure</u>

We'll assemble the global structural stiffness matrix for the structure in Fig. 2, then solve for displacements, reactions, and internal forces. Note that we have labeled the unfrozen D.O.F. as r1 - r5, then the frozen D.O.F. as r6 - r9. R1 - R5 will represent the external forces, in this case R1=5, R2=R3=R4=R5=0. R6 - R9 represent the reactions.

The process is simply to create Ke for each of the two members using Eq. 12, then to add them by keeping track of how the local degrees of freedom correspond to the global degrees of freedom.

Note that this problem is *indeterminate*, but this does not affect our procedure or solution!

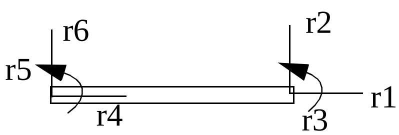


Fig. 3. Element 1

<u>Element 1:</u> the element labeled 1 connects nodes 1 and 2, and global D.O.F. r1 - r6. Finding Lx and Ly and applying Eq. 12. :

Lx = change in x / L = (20 - 0)/20 = 1

Ly = change in y / L = (0 - 0)/20 = 0

$$\begin{array}{c} \\ Ke1 = \end{array} \begin{array}{cccccc} r4 & r6 & r5 & r1 & r2 & r3 \end{array}$$

$$Ke1 = \begin{bmatrix} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7e3 & 0 & -1510.4 & 120.83e3 \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83e3 & 0 & -1510.4 & 241.7e3 \end{bmatrix} \begin{array}{c} r4 \\ r6 \\ r5 \\ r1 \\ r2 \\ r3 \end{array} \qquad (13)$$

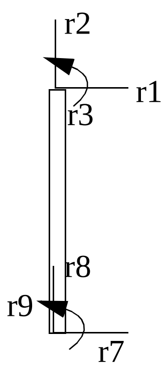


Fig. 4. Element 2

<u>Element 2:</u> This element connects nodes 2 and 3, and global D.O.F. r1-r3, r7-r9 :

Lx = change in x / L = (20 - 20)/20 = 0

Ly = change in x / L = (-20 - 0)/20 = -1

this gives us $Ke2$ through Eq. 12 as

$$\begin{array}{cccccc} r1 & r2 & r3 & r7 & r8 & r9 \end{array}$$

$$Ke2 = \begin{bmatrix} 12.6 & 0 & 1510.4 & -12.6 & 0 & 1510.4 \\ 0 & 1208.3 & 0 & 0 & -1208.3 & 0 \\ 1510.4 & 0 & 241.7e3 & -1510.4 & 0 & 120.83e3 \\ -12.6 & 0 & -1510.4 & 12.6 & 0 & -1510.4 \\ 0 & -1208.3 & 0 & 0 & 1208.3 & 0 \\ 1510.4 & 0 & 120.83e3 & -1510.4 & 0 & 241.7e3 \end{bmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \\ r7 \\ r8 \\ r9 \end{array} \qquad (14)$$

$A=10\ in^2$

$I = 500\ in^4$

$E = 29000\ ksi$

Note that we have listed above and to the right of each element global stiffness matrix the global displacements that correspond to the six local degrees of freedom.

Now we have all element stiffness matrices. We simply set up the global structural stiffness matrix as a 9x9 with all zeros initially, and add in the element matrices.

$$KG = Ke1 + Ke2 \qquad (15)$$

After assembly, defining known forces and displacements, and partitioning as we did for the truss example, we have (confirm this yourself)

$$\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline R6 \\ R7 \\ R8 \\ R9 \end{bmatrix} = \left[\begin{array}{ccccc|cccc} 1220.9 & 0 & 1510.4 & -1208.3 & 0 & 0 & -12.6 & 0 & 1510.4 \\ 0 & 1220.9 & -1510.4 & 0 & -1510.4 & -12.6 & 0 & -1208.3 & 0 \\ 1510.4 & -1510.4 & 483.4e3 & 0 & 120.83e3 & 1510.4 & -1510.4 & 0 & 120.83e3 \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1510.4 & 120.83e3 & 0 & 241.7e3 & 1510.4 & 0 & 0 & 0 \\ \hline 0 & -12.6 & 1510.4 & 0 & 1510.4 & 12.6 & 0 & 0 & 0 \\ -12.6 & 0 & -1510.4 & 0 & 0 & 0 & 12.6 & 0 & -1510.4 \\ 0 & -1208.3 & 0 & 0 & 0 & 0 & 0 & 1208.3 & 0 \\ 1510.4 & 0 & 120.83e3 & 0 & 0 & 0 & -1510.4 & 0 & 241.7e3 \end{array}\right] \begin{bmatrix} r1 \\ r2 \\ r3 \\ r4 \\ r5 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (16)$$

Note that we get all 5 known forces at the top of the force vector, and the last 4 components in the displacement vector are known because we labeled the unfrozen degrees of freedom first.

7. The Solution Procedure

<u>The knowns</u>:
Rk = R1 - R5 - external forces
rk = r6 - r9 - frozen displacements (all = 0)

<u>The unknowns:</u>
Ru = R6 - R9 - reactions
ru = r1 - r5 - displacements
rewrite Eq. 16 in shorthand as:

$$\begin{bmatrix} Rk \\ \hline Ru \end{bmatrix} = \left[\begin{array}{c|c} K11 & K12 \\ \hline K21 & K22 \end{array}\right] \begin{bmatrix} ru \\ \hline rk \end{bmatrix} \qquad (17)$$

now expanding we get...(make sure you can follow this step)

$$Rk = K11*ru + K12*rk \qquad (18)$$

$$Ru = K21*ru + K22*rk \qquad (19)$$

We can solve the system in Eq. 18 (which is 5 equations and 5 unknowns) for ru, then solve the system in Eq. 19 (a 4 equation, 4 unknown system). The solutions are below:

$$\begin{bmatrix} r1 \\ r2 \\ r3 \\ r4 \\ r5 \end{bmatrix} = \begin{bmatrix} 0.696 \text{ in.} \\ -1.55e-3 \text{ in.} \\ -2.488e-3 \text{ rad} \\ 0.696 \text{ in.} \\ 1.234e-3 \text{ rad} \end{bmatrix}, \qquad \begin{bmatrix} R6 \\ R7 \\ R8 \\ R9 \end{bmatrix} = \begin{bmatrix} -1.87 \text{ k} \\ -5.00 \text{ k} \\ 1.87 \text{ k} \\ 750 \text{ k*in.} \end{bmatrix}$$

Now force recovery is given by applying Eq. 8 $S = k*a*r$ for each element, recalling that we are using the local element stiffness matrices given in Eqs. 13 and 14, and being careful to apply the correct displacements r by referring to Figures 3 and 4.

Element 1:

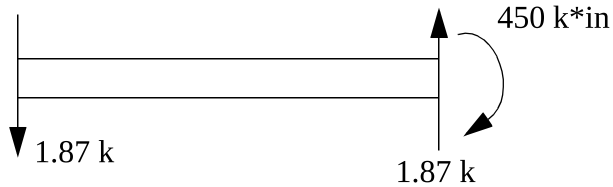
$$
\begin{matrix} & r4 & r6 & r5 & r1 & r2 & r3 \end{matrix}
$$

$$
S = \begin{bmatrix} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7e3 & 0 & -1510.4 & 120.83e3 \\ -1208.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83e3 & 0 & -1510.4 & 241.7e3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.696 \text{ in.} \\ 0 \\ 1.234e-3 \text{ rad} \\ 0.696 \text{ in.} \\ -1,55e-3 \text{ rad} \\ -2.488e-3 \text{ rad} \end{bmatrix} \begin{matrix} r4 \\ r6 \\ r5 \\ r1 \\ r2 \\ r3 \end{matrix} \quad (20)
$$

which gives us

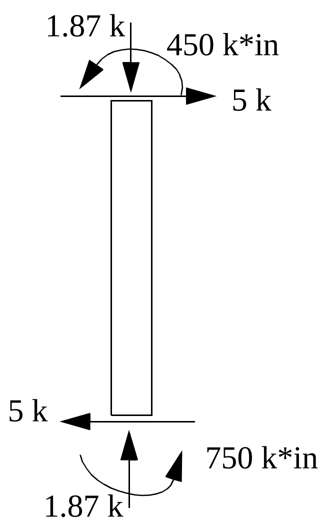
$$
\begin{bmatrix} S4 \\ S6 \\ S5 \\ S1 \\ S2 \\ S3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.87 \text{ k} \\ 0 \\ 0 \\ 1.87 \text{ k} \\ -450 \text{ k*in.} \end{bmatrix} \quad (21)
$$

which is just



Element 2:

The above procedure is applied to element 2, and gives (confirm this):



What should be obvious now is that although the procedure is quite systematic, it gets quite lengthy to solve by hand. Not only putting together the stiffness matrix, but the solution of Eq. 18 using Gaussian Elimination is time consuming. However, given the systematic nature of the method, it is easily programmed. This is why matrix methods are universally chosen as the method of choice in engineering analysis software.

What do we do about hinges, mixed beam and trusses, and beams with no axial displacement?...