Virtual Work Derivation

Objectives:

1. Calculate the internal virtual work for a truss
2. Calculate the internal virtual work for a beam
3. Calculate the external virtual work

Derivation Assumptions

Virtual Work will handle all of the requirements needed for this course.

Assumes that the structure is at a stable equilibrium position. If a small perturbation (movement) from this position is introduced, the structure will return to its equilibrium position. In addition, the work done to move the structure (in an energy sense) is zero.

The work of the applied loads moving through the perturbation is stored in the structure as elastic energy.

Two types of work:

External work is the work caused by the loads acting on the structure
Internal work is a potential energy type work that is stored in beams, columns and trusses as a result of stretching and bending.

The equation for the virtual work balance can be written as:

$$\delta W_e = \delta W_i$$

Definitions

R is a vector of externally applied concentrated loads.
Individual terms in the vector **R** can be identified as R_i.

r is a vector of global external displacements for a structure.

The external virtual work done on a structure can be calculated by:

$$\delta W_e = \sum_j \overline{R_j} * r_j = \sum_j R_j * \overline{r_j} = \overline{\mathrm{R}}^T * \mathrm{r} = \overline{\mathrm{r}}^T * \mathrm{R}$$

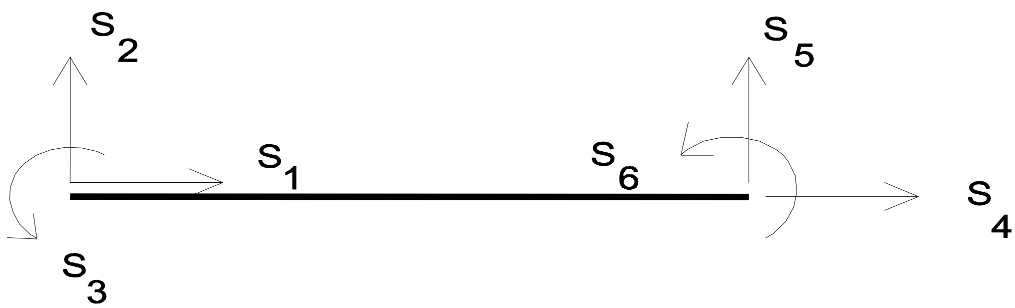
where the bar over the vector indicates that the quantity is a virtual quantity.

Or

$\int R(x) * r(x)\, dx$

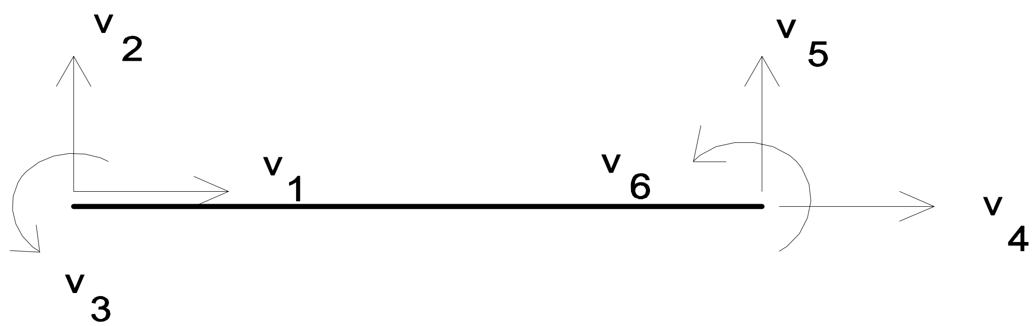
where R(x) and r(x) are continuous functions of load and displacement

S is the vector of internal forces. They are concentrated values at particular points on a member. For example, the end moments on a beam are internal forces.



Definition of Internal Forces

v is the vector of internal displacements. They are concentrated values corresponding to the internal loads. For example, the end rotations of a beam which correspond to the end moments.



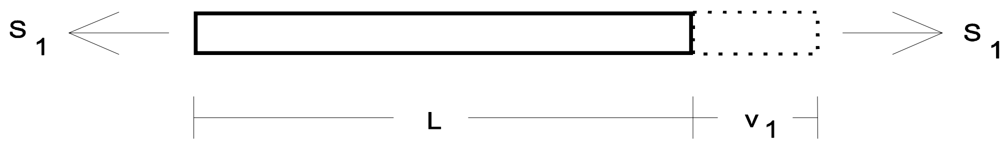
Definition of Internal Displacements

The internal *virtual* work is:

$$\delta W_i = \sum_j \overline{S_j} * v_j = \sum_j S_j * \overline{v_j} = \overline{S}^T * v = \overline{v}^T * S$$

Internal Work for Trusses

Consider a bar subjected to an axial load S_1. It has a displacement v_1 at the end of the bar.



Deformed Truss Bar

Clearly, the work (force times displacement) is:

$$W_i = S_1 * v_1$$

From mechanics, we know that the displacement of an axial member is related to the applied force by:

Solving for v_1 we get:

Substituting into the work equation we have:

$$W_i = \frac{S_1 S_1 L}{A E}$$

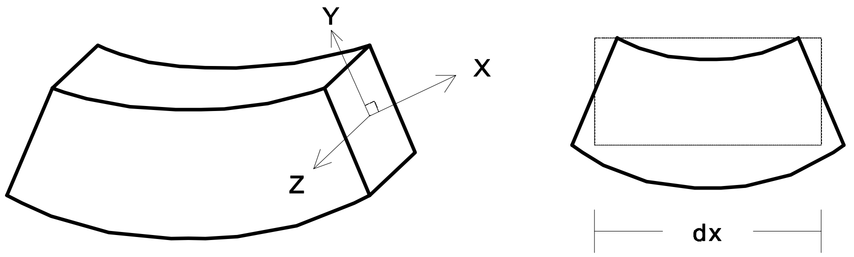
For the virtual work, either S_1 or v_1 need to be made virtual.

$$\delta W_i = v_1 \overline{S_1} = \overline{v_1} S_1 = \frac{\overline{S_1} S_1 L}{A E}$$

Internal Work for Beam Elements

The stored energy in a beam can only come from bending (ignoring axial and shear work).

The internal work for any continuum can be measured as strain energy. Strain energy is the product of stress times strain.



Deformed Beam Differential Element

Pure Bending of Beam Element

The internal work, equal to the strain energy of the beam is:

$$W_i = \int_{length} \int_{area} \sigma \, \varepsilon \, dA \, dx = \int_{length} \int_{width} \int_{height} \sigma \, \varepsilon \, dy \, dz \, dx$$

Again from mechanics we know the following:

and

where M is the moment on the face, y is the distance from the centroid, I is the moment of inertia and E is Young=s modulus.

Substituting we get:

$$W_i = \int_{length} \int_{area} \frac{M\,y}{I} \frac{M\,y}{E\,I} \, dA \, dx$$

The moment, M, moment of inertia, I, and Young=s Modulus, E, are all constant for a cross section and can be removed from the integration on the area, dy and dz. This leaves:

$$W_i = \int_{length} \frac{M\ M}{E\ I^2} \int_{area} y^2\, dy\, dz\, dx$$

But, the integral of y^2 is the definition of the moment of inertia.

This final equation is of the form:

$$W_i = \int_{length} \frac{M\ M}{E\ I}\, dx$$

For virtual work, one of the moments needs to be made a virtual quantity:

$$\delta W_i = \int_{length} \frac{\overline{M}\ M}{E\ I}\, dx$$

Note that $\frac{M}{EI}$ is defined as the curvature (ψ). Therefore, beam work can be thought of as moment times curvature integrated over the length of the beam.

$$\delta W_i = \int_{length} \overline{M}\ \psi\, dx$$

This is equivalent to moment times rotation.