Derivation of Stiffness by Direct Stiffness

Objectives:

1) Calculate the 2-D beam stiffness using the method "by definition"
2) Write the global stiffness equation

The **a** matrix converts the deformations from the global to the local coordinate system. In virtual form we have:

$$\bar{v} = a\,\bar{r}$$

The external virtual work is:

$$\delta W_e = \bar{r}^T R$$

The internal virtual work is the sum of the work from all the elements in a structure.

$$\delta W_i = \sum_{\text{all elements}} \bar{v}^T S$$

Relating the element displacements to the element forces we have:

$$S = K_e\, v$$

where $\mathbf{K_e}$ is the element stiffness matrix that relates the element DOF to the element forces. Substituting this into the δW_i:

$$\delta W_i = \sum_{\text{all elements}} \bar{v}^T K_e\, v$$

Substituting for **v** and equating internal to external work we have:

$$\bar{r}^T R = \sum_{\text{all elements}} \bar{r}^T a^T K_e\, a\, r$$

Since $\bar{r}^T$ is both arbitrary and small, we can cancel it from both sides of the equation.

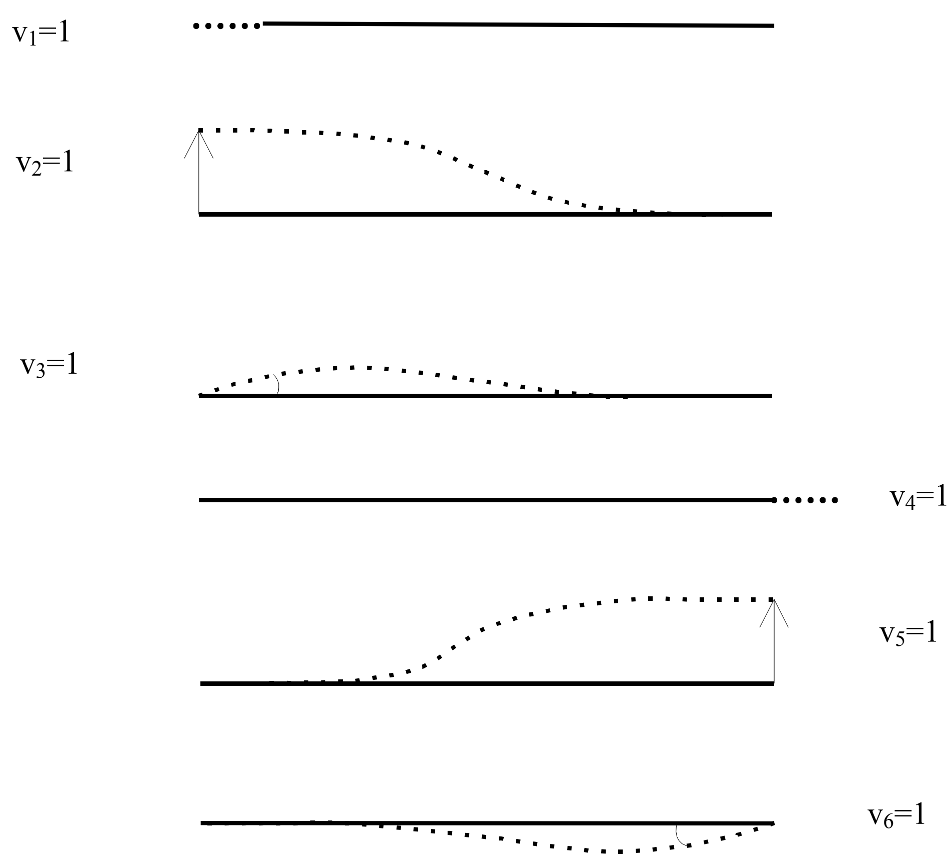
$$R = \left[\sum_{\text{all elements}} a^T K_e\, a \right] r$$

The global stiffness must be:

$$K_g = \left[\sum_{\text{all elements}} a^T K_e \, a \right]$$

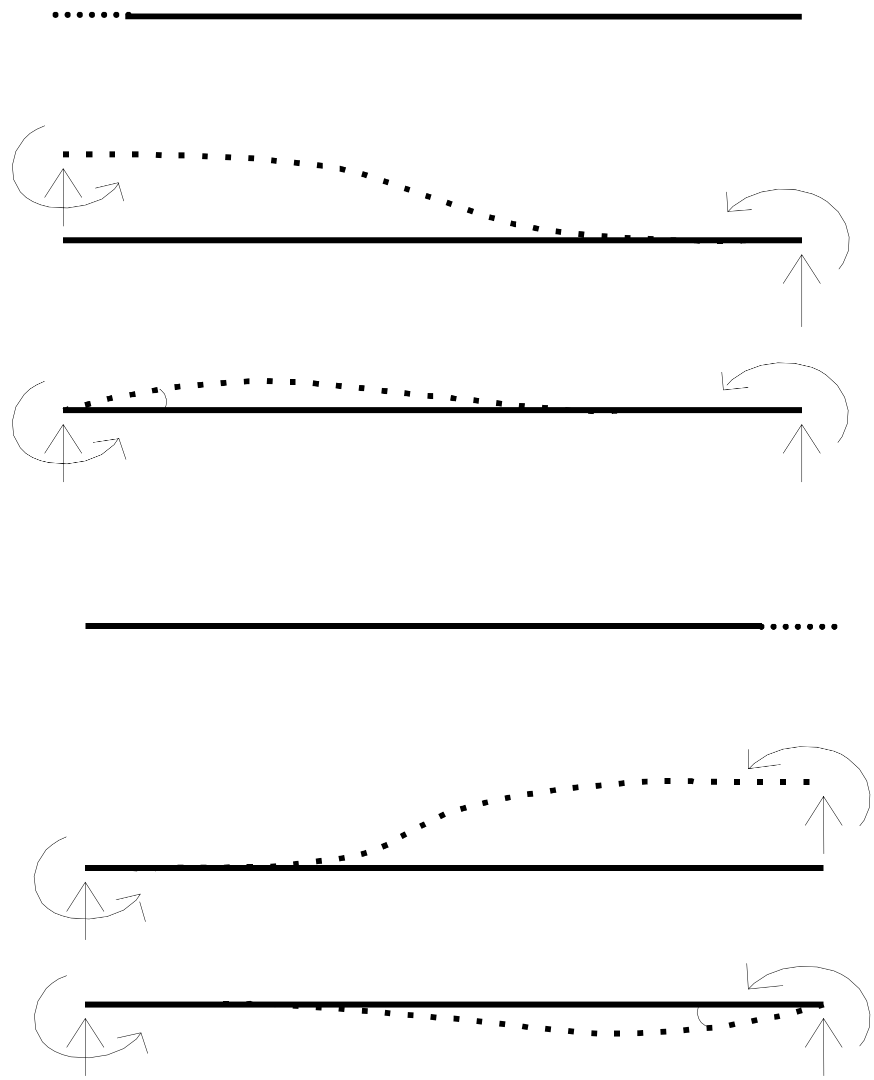
2-D Beam Element Stiffness Derivation

We need the $\mathbf{K_e}$ matrix. Use the definition.



Deformed Shapes for Element Stiffness Derivation

The moments come directly from slope deflection. (Just like were used in stiffness by definition.) The shear forces can then be determined from statics.



Forces Required for Unit Deformations

Putting these forces into the correct places in the stiffness matrix we have:

$$K_e = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12\,EI}{L^3} & \frac{6\,EI}{L^2} & 0 & -\frac{12\,EI}{L^3} & \frac{6\,EI}{L^2} \\ 0 & \frac{6\,EI}{L^2} & \frac{4\,EI}{L} & 0 & -\frac{6\,EI}{L^2} & \frac{2\,EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12\,EI}{L^3} & -\frac{6\,EI}{L^2} & 0 & \frac{12\,EI}{L^3} & -\frac{6\,EI}{L^2} \\ 0 & \frac{6\,EI}{L^2} & \frac{2\,EI}{L} & 0 & -\frac{6\,EI}{L^2} & \frac{4\,EI}{L} \end{bmatrix}$$