Basic Linear Algebra

Objectives:

1. Write a matrix and know it's size
2. Add matrices
3. Subtract matrices
4. Multiply matrices
5. Define a matrix transpose
6. Define an Identity matrix
7. Define a symmetric matrix

Matrix methods are a subset of linear algebra.

A matrix has a size defined by its number of rows and columns.

$$A = \begin{bmatrix} 1.2 & 2.3 \\ 5.4 & 1.0 \\ 0.0 & 0.2 \end{bmatrix}$$

The size of matrix **A** is 3 rows by 2 columns.

In common language it is said, "matrix **A** is 3 by 2".

Any term in the matrix is referred to by its row and column position. The row position is always given first. (i.e. **A**(2,2) = 1.0)

The diagonal of a matrix is defined as all terms where the row number and the column number are the same. (i.e. **A**(1,1) and **A**(2,2)).

Matrices can be added. Matrices **MUST** be the same size.

The resulting values is the sum of the corresponding terms

$$B = \begin{bmatrix} 1.2 & 9.2 \\ 3.4 & 2.1 \\ 3.1 & 6.5 \end{bmatrix}$$

Then: $A + B = C$ Gives the result: $C = \begin{bmatrix} 0.0 & \\ & 1.1 \\ 3.1 & \end{bmatrix}$

Subtraction is handled in a similar way.

$D = A - B$ gives the result $D = \begin{bmatrix} 2.4 & \\ & \\ 3.1 & 6.7 \end{bmatrix}$

Matrices can be multiplied, but not divided.

Matrix multiplication is **NOT** commutative. (it is order dependent).

The terms of the result consist of multiplying a row of the first matrix times a column of the second matrix.

$G = A * F$ $F = \begin{bmatrix} 3.4 & 2.1 \\ 7.3 & 1.0 \end{bmatrix}$

The 1,1 term (row one, column one) of **G** is:

$G(1,1) = 1.2 * 3.4 + (2.3) * 7.3 = 12.71$

The location where the result goes is dictated by the row and column number being multiplied.

The result **G**(3,2) comes from multiplying row 3 of **A** times column 2 of **F**. The final matrix **G** is:

$$G = \begin{bmatrix} 12.71 & 4.82 \\ & \\ 1.46 & \end{bmatrix}$$

Matrix sizes must match:

$$\begin{matrix} G & = & A & * & F \\ (N \times L) & & (N \times M) & & (M \times L) \end{matrix}$$

Must be same.

Multiplication is associative and distributive.

$A * B * C = (A * B) * C = A * (B * C)$ and $(A + B) * C = A * C + B * C$

There is a multiplicative identity matrix, **I**. Defined as a matrix that has ones on the diagonal and zeros everywhere else (off diagonal terms). The identity matrix is always square (same number of rows as columns). For example, a 3 by 3 identity matrix looks like:

$$I = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

The identity matrix has the property:

$$A * I = I * A = A$$

The inverse of a matrix is the matrix that when multiplied times the original matrix will give the identity matrix. The inverse of matrix **A** is designated as $\mathbf{A}^{-1}$.

$$A * A^{1} = A^{1} * A = I$$

A matrix MUST be square to be inverted. Not all matrices can be inverted. An invertible matrix is called non-singular or invertible. If it *cannot* be inverted, then it is called singular.

Matrix inversion is very in the solution of equations.

$$\begin{aligned} 5\,x_1 \;\; 4\,x_2 + \;\; x_3 \;\;\;\;\;\; &= 0 \\ 4\,x_1 + 6\,x_2 \;\; 4\,x_3 + \;\; x_4 &= 1 \\ x_1 \;\; 4\,x_2 + 6\,x_3 \;\; 4\,x_4 &= 0 \\ x_2 \;\; 4\,x_3 + 5\,x_4 &= 2 \end{aligned}$$

Is represented by:

$$A = \begin{bmatrix} 5 & 4 & 1 & 0 \\ 4 & 6 & 4 & 1 \\ 1 & 4 & 6 & 4 \\ 0 & 1 & 4 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

The set of equations is represented by:

$$A\,x = b$$

Using matrix algebra;.

$$A^{-1} A x = A^{-1} b$$

$$I\ x = A^{-1} b$$

$$x = A^{-1} b$$

In practice the inverse is never calculated, a solution process is used.

There are two types of solution methods used in practice: 1) Direct and 2) Iterative. Direct solutions can be thought of as variations of the Gauss Elimination method. The iterative schemes can be thought of as variations of the Gauss-Sidel solution technique.

The transpose is defined as switching the rows with the columns. A superscript T designates the transpose. For example, the transpose of matrix **H** is $\mathbf{H}^{\mathbf{T}}$:

$$\mathrm{H} = \begin{bmatrix} 1.2 & 5.4 & 3.3 & 0.0 \\ 4.3 & 1.1 & 3.1 & 7.8 \\ 1.0 & 0.0 & 8.2 & 1.9 \end{bmatrix} \qquad \mathrm{H}^{\mathrm{T}} = \begin{bmatrix} 1.2 & 4.3 & 1.0 \\ 5.4 & 1.1 & 0.0 \\ & & \\ 0.0 & 7.8 & 1.9 \end{bmatrix}$$

(3 by 4) (4 by 3)

The transpose can also be taken of a set of multiplied matrices by reversing the order of the multiplications and individually transposing the matrices.

$$(\ H * A * D\ )^T = D^T A^T H^T$$

A matrix is symmetric when the rows are equal to the corresponding column. In other words, the matrix is equal to its transpose.

$$\mathrm{M} = \begin{bmatrix} 6.0 & 2.0 & 1.0 & 0.0 & 1.0 & 0.0 \\ 2.0 & 5.0 & 1.0 & 2.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 6.0 & 1.0 & 2.0 & 1.0 \\ 0.0 & 2.0 & 1.0 & 4.0 & 1.0 & 2.0 \\ 1.0 & 0.0 & 2.0 & 1.0 & 5.0 & 1.0 \\ 0.0 & 0.0 & 1.0 & 2.0 & 1.0 & 5.0 \end{bmatrix}$$